A Gauss-Bonnet formula for closed semi-algebraic sets.

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A GAUSS-BONNET FORMULA FOR CLOSED SEMI-ALGEBRAIC SETS

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Abstract. We prove a Gauss-Bonnet formula for closed semi-algebraic sets.

1. Introduction

Let S be a smooth compact surface embedded in $\mathbb{R}^3$. We know from Jordan-Brouwer Separation Theorem that S is orientable as the boundary of a smooth compact oriented manifold of dimension 3. This means that on S there exists a smooth unitary normal vector field $\vec{n}$. The map $g : S \to S^2$ defined by $g(x) = \vec{n}(x)$ is called the Gauss map of S and its Jacobian $J_g(x) := k(x)$ is called the (Gaussian) curvature of S at x. The curvature $k(x)$ describes the local aspect of S at the point x : its absolute value measures how S is curved at x and its sign serves to distinguish between local convex appearance and local saddlelike appearance. Although it is defined using an embedding and the choice of an unitary normal vector field, the curvature is an intrinsic concept of the surface and is invariant by local isometries (this is the Gauss Theorema Egregium). It is not preserved by homeomorphisms because of its obvious geometric nature.

However the famous Gauss-Bonnet theorem asserts that the global integral of the curvature on the surface is a topological invariant, namely 2π times the Euler-Poincaré characteristic of S. This result is one of the most important theorem in differential geometry. Its first generalization was obtained by Hopf [Ho1] for an even-dimensional hypersurface $M \subset \mathbb{R}^n$. As in the case of a surface embedded in $\mathbb{R}^3$, the curvature k is defined as the Jacobian of the Gauss map $g : M \to S^n$. If we denote by dx the volume element of M, then one has :

$$\int_M k(x)dx = \frac{1}{2}\mathrm{Vol}(S^{n-1}) \cdot \chi(M).$$

Hopf's proof is essentially topological and makes use of very few differential geometric arguments. Indeed, if dv denotes the volume element of the sphere S^{n-1}, then, by a classical change of variables, one has :

$$\int_M k(x)dx = \int_M g^*(dv) = \mathrm{Vol}(S^{n-1}) \cdot \deg(g).$$

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Using the Poincaré-Hopf Theorem, one can prove that $\deg(g) = \frac{1}{2}\chi(M)$. In [Ho2], Hopf asked for generalizations of his result and also for intrinsic proofs, since the curvature of a surface is intrinsic. The first question was answered by Allendoerfer [Al] and Fenchel [Fe] for a compact n-dimensional manifold M embedded in $\mathbb{R}^N$. Let us present briefly Fenchel's results. First he defined the Lipschitz-Killing curvature at a point x in M in the following way. If v is a unit normal vector to M at x and if π_v denotes the orthogonal projection from M to the $(n+1)$-dimensional vector space spanned by T_xM and v, then the image of this projection is a hypersurface non-singular at x and oriented by v. Let us denote by $k(x, \pi_v(M))$ its Gaussian curvature at x. The Lipschitz-Killing curvature at x is :

$$LK(x) = C(N,n) \int_{NU_xM} k(x, \pi_v(M)) dv,$$

where NU_xM is the unitary normal space of M at x and where $C(N,n)$ depends only on N and n. Fenchel proved that :

$$\int_M LK(x)dx = \frac{1}{2}\mathrm{Vol}(S^{N-1}) \cdot \chi(M).$$

The second question was answered partially by Allendoerfer and Weil [AW] and then completely by Chern [Ch].

Let us focus now on the extrinsic version of the Gauss-Bonnet formula. We have seen above that if M is a hypersurface embedded in $\mathbb{R}^n$ then :

$$\int_M k(x)dx = \mathrm{Vol}(S^{n-1}) \cdot \deg(g).$$

This formula is actually a special case of the following result. Let U be an open subset of M. For almost all $L \in G(n,1)$, the Grassmann manifold of lines in $\mathbb{R}^n$, the orthogonal projection $p_L : U \to L$ is a Morse function, which admits a finite number of critical points $p_1^L, \ldots, p_{r_L}^L$. Let $\mu(U,L)$ be defined by $\mu(U,L) = \sum_{j=1}^{r_L} \deg(g, p_j^L)$, where $\deg(g, p_j^L)$ is the local topological degree of the Gauss map at p_j^L. The exchange principle (see [La], [LS]) states that :

$$\int_U k(x)dx = \int_{G(n,1)} \mu(U,L)dL.$$

When $U = M$, using the fact that $\mu(M,L) = 2\deg(g)$ generically, we recover Hopf's formula. Similar formulas for $\int |k|$ were proved and used by Milnor [Miln1], Chern and Lashof [CL] and Kuiper [Ku1]. Langevin [La] also established such formulas for $\int_U LK$ and $\int_U |LK|$ where U is an open subset of a n-dimensional manifold M embedded in $\mathbb{R}^N$, which enabled him to give a new proof of Fenchel's theorem. Exchange principles are interesting because they provide us with a link between the differential geometry and the differential topology. They also offer a new approach in the study of differential geometric properties of smooth manifolds, which can be adopted in order to obtain Gauss-Bonnet formulas for non-smooth objects.

This is exactly the point of view of Banchoff ([Ba1], [Ba2]) in his study of embedded polyhedra. Let us describe briefly this work. Let M be an embedded polyhedron of dimension k in $\mathbb{R}^n$. A linear map $\xi : \mathbb{R}^n \to \mathbb{R}$ is general for M if $\xi(a) \neq \xi(b)$ whenever a and b are the vertices of a one-dimensional cell in M. If ξ is general for M and D^r is a r-dimensional cell of M, we may define an indicator function :

$$A(D^r, a, \xi) = \begin{cases} 1 \text{ if } a \in D^r \text{ and } \xi(a) \geq \xi(x) \text{ for all } x \in D^r, \\ 0 \text{ otherwise,} \end{cases}$$

and then, the index of a with respect to ξ :

$$\alpha(a, \xi) = \sum_{r=1}^{k} (-1)^r \sum_{D^r \subset M} A(D^r, a, \xi).$$

If Ω is an open set of M, the index of Ω with respect to ξ is $\alpha(\Omega, \xi) = \sum_{a \in \Omega} \alpha(a, \xi)$. It is defined and finite for almost all ξ. The curvature of Ω is

$$K(\Omega) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \alpha(\Omega, v^*) dv,$$

where for $v \in S^{n-1}$, v^* is the linear form $v^*(x) = -\langle v, x \rangle$. Banchoff proved that $K(\Omega)$ is an intrinsic metric invariant and that $K(M) = \chi(M)$. This last equality is the Gauss-Bonnet theorem for embedded polyhedra. This method was generalized by Kuiper to topological manifolds in [Ku2].

In [BK], Bröcker and Küppe developped integral geometry for tame sets. These sets are closed Whitney-stratified sets that satisfy two extra conditions (see Definition 2.9). They include piecewise linear sets, semi-algebraic sets, subanalytic sets, sets in an o-minimal structure, $\mathcal{X}$ sets [Sh], manifolds with boundary, Riemannian polyhedra. Their main tool is Morse stratified theory introduced by Goresky and Mac-Pherson [G.M-P]. Their results contain of course a Gauss-Bonnet theorem. If v is a vector in S^{n-1}, let us denote by v^* the linear form $v^*(x) = -\langle v, x \rangle$. Let X be a tame set. For almost all v in S^{n-1}, the function $v^*_{|X} : X \to \mathbb{R}$ is a Morse function (in the sense of Goresky and Mac-Pherson). To each point x in X, one can assign an index $\alpha(X, v^*, x)$ in the following way : if x is a critical point of $v^*_{|X}$ then $\alpha(X, v^*, x)$ is the Morse index of $v^*_{|X}$ at x (see Definition 2.1), if x is not a critical point of $v^*_{|X}$ then $\alpha(X, v^*, x) = 0$. Then one can define a curvature measure on Borel sets U of X by :

$$\lambda_0(U) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in U} \alpha(X, v^*, x) dv.$$

If X is a compact tame set, then one has [BK, Proposition 4.1] :

$$\lambda_0(X) = \chi(X).$$

Bröcker and Küppe also showed that, if we restrict ourselves to an o-minimal structure (resp. an $\mathcal{X}$-system), $\lambda(U)$ is invariant by definable isometries (resp. $\mathcal{X}$-isometries). One should mention that Fu [Fu1] also developped integral geometry for subanalytic sets using the construction of normal cycles.

In [Dut3], we established a Gauss-Bonnet formula for a smooth fiber of a nonproper real polynomial of $\mathbb{R}^n$. Our aim here is to give such a formula for a closed semi-algebraic set. Let $X \subset \mathbb{R}^n$ be a closed (not necessarily compact) semi-algebraic set. Let $(K_R)_{R>0}$ be an exhaustive sequence of compact borelian sets of X ; this means that $\cup_{R>0} K_R = X$ and that $K_R \subseteq K_{R'}$ if $R \leq R'$. The limit $\lim_{R\to+\infty} \lambda_0(X \cap K_R)$ is finite and independent on the choice of the exhaustive sequence. We define $\lambda_0(X)$ to be this limit and we prove (Theorem 4.5) :

$$\lambda_0(X) = \chi(W) - \frac{1}{2}\chi(Lk^\infty(X)) \\ -\frac{1}{2\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} \chi(Lk^\infty(X \cap \{v^* = 0\}))dv,$$

where $Lk^\infty(X) = X \cap S_R^{n-1}$, $R \gg 1$, is the link at infinity of X. Let us explain shortly the different steps of our proof. First observe that $\lambda_0(X) = \lim_{R\to+\infty} \lambda_0(X \cap \mathring{B}_R^n)$ where $\mathring{B}_R^n$ is the interior of the the euclidian ball of radius R. For each R, let $f_R : \mathbb{R}^n \to \mathbb{R}$ be an approximating function from outside for the semi-algebraic set $X \cap B_{2R}^n$ (see Definition 2.12). In particular, f_R is non-negative and $X \cap B_{2R}^n = f_R^{-1}(0)$. Furthermore, we have ([BR], Theorem 6.2) :

$$\lambda_0(X \cap \mathring{B}_R^n) = \lim_{t\to 0} \lambda_0(\{f_R \leq t\} \cap \mathring{B}_R^n).$$

Hence we are lead to compute :

$$\lim_{R\to+\infty}\lim_{t\to 0} \lambda_0(\{f_R \leq t\} \cap \mathring{B}_R^n) = \\ \lim_{R\to+\infty}\lim_{t\to 0} \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} \sum_{x\in\{f_R\leq t\}\cap\mathring{B}_R^n} \alpha(\{f_R \leq t\}, v^*, x)dv.$$

But, for R sufficiently big and t sufficiently small, $\{f_R \leq t\} \cap B_R^n$ is a manifold with corners. Applying Morse theory for manifolds with corners, we find that for $v \in S^{n-1}$:

$$\chi(\{f_R \leq t\} \cap B_R^n) = \sum_{x\in\{f_R\leq t\}\cap\mathring{B}_R^n} \alpha(\{f_R \leq t\}, v^*, x) + \Lambda^{t,R,v},$$

where $\Lambda^{t,R,v}$ is the contribution of the critical points of $v^*_{|\{f_r\leq t\}\cap S_R^{n-1}}$. A careful analysis of the behaviour of these last critical points as R tends to infinity and t tends to 0, based on a study of some polar varieties, enables us to show that :

$$\lim_{R\to+\infty}\lim_{t\to 0} \Lambda^{t,R,v} + \Lambda^{t,R,-v} = \chi(Lk^\infty(X)) + \chi(Lk^\infty(X \cap \{v^* = 0\})).$$

We conclude using the two following facts :

$$\lim_{R\to+\infty}\lim_{t\to 0}\chi(\{f_R\le t\}\cap B_R^n)=\chi(X),$$

and

$$\int_{S^{n-1}}\lim_{R\to+\infty}\lim_{t\to 0}\Lambda^{t,R,v}dv=\frac{1}{2}\int_{S^{n-1}}\lim_{R\to+\infty}\lim_{t\to 0}[\Lambda^{t,R,v}+\Lambda^{t,R,-v}]dv.$$

If M is a compact boundaryless manifold of dimension n in $\mathbb{R}^N$, let M_ε denote the closed ε-neighborhood of M (i.e the set of all x in $\mathbb{R}^N$ such that $\|x-y\|\le\varepsilon$ for some $y\in M$). For ε small enough, M_ε is a smooth N-dimensional manifold with boundary, diffeomorphic to a tubular neighborhood of M. Its boundary ∂M_ε is a smooth hypersurface, let k_ε be its curvature and let g_ε be the Gauss mapping $\partial M_\varepsilon\to S^{N-1}$. One has :

$$\int_{M_\varepsilon}k_\varepsilon(x)dx=\mathrm{Vol}(S^{N-1})\cdot\deg(g_\varepsilon).$$

Since $\deg(g_\varepsilon)=\chi(X)$ if ε is sufficiently small (see [Miln2], p38), we obtain the following equality (first observed by Fenchel [Fe] and Allendoerfer [Al] for even dimensional manifolds) :

$$\lim_{\varepsilon\to 0}\int_{M_\varepsilon}k_\varepsilon(x)dx=\mathrm{Vol}(S^{N-1})\cdot\chi(X).$$

Our second main result is a semi-algebraic version of this equality. If X is a closed semi-algebraic set in $\mathbb{R}^n$, we showed in [Dut4] that there exists a non-negative semi-algebraic function f of class C^2 such that $X=f^{-1}(0)$ and such that for $\delta>0$ sufficiently small, the inclusion $X\subset f^{-1}([0,\delta])$ is a retraction. We called f an approaching function for X and $f^{-1}([0,\delta]]$ a semi-algebraic approaching neighborhood of X. This last set can be considered as a semi-algebraic analogue of the tubular neighborhood of differential geometry. Its boundary $f^{-1}(\delta)$ is a C^2 semi-algebraic hypersurface. If k_δ denotes its curvature and if $(K_R^\delta)_{R>0}$ is an exhaustive sequence of compact sets in $f^{-1}(\delta)$, then the limit $\lim_{R\to+\infty}\int_{f^{-1}(\delta)\cap K_R^\delta}k_\delta dx$ is finite and does not depend on the sequence chosen. Using the results of [Dut3] and [Dut4], we prove that there exists an approaching function f for X such that (Theorem 5.5) :

$$\lim_{\delta\to 0^+}\int_{f^{-1}(\delta)}k_\delta dx=\mathrm{Vol}(S^{n-1})\cdot\chi(W)-\frac{1}{2}\mathrm{Vol}(S^{n-1})\cdot\chi(Lk^\infty(X))$$
$$-\frac{1}{2}\int_{S^{n-1}}\chi(Lk^\infty(X\cap\{v^*=0\}))dv.$$

As a direct corollary, we get (Corollary 5.6) :

$$\mathrm{Vol}(S^{n-1})\cdot\lambda_0(X)=\lim_{\delta\to 0^+}\int_{f^{-1}(\delta)}k_\delta dx.$$

The paper is organized as follows. Section 2 is a summary of the results about Morse stratified theory and tame stratified sets that we will use in the sequel. In Section 3, we study the critical points of $v^*_{|\{f_R \le t\} \cap S_R^{n-1}}$ for v generic. In Section 4, we prove the first version of the Gauss-Bonnet theorem for closed semi-algebraic sets. Section 5 is independent on the two previous ones and deals with the proof of the second version of the Gauss-Bonnet theorem, i.e the Gauss-Bonnet theorem "by approximation".

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2. Morse stratified theory and tame stratified sets

This section is essentially a presentation of results contained in [BK]. Let M be a C^3 riemannian manifold of dimension n. Let X be a Whitney-stratified set of M. Let x be a point in X and let $S(x)$ be the stratum that contains x. A generalized tangent space at x is a limit of a sequence of tangent spaces $(T_{y_k} S_1)_{k \in \mathbb{N}}$, where S_1 is a stratum distinct from $S(x)$ such that $x \in \overline{S_1}$ and $(y_k)_{k \in \mathbb{N}}$ is a sequence of points in S_1 tending to x. The point x is a critical point of a function $f : X \to \mathbb{R}$ if it is a critical point of $f_{|S(x)}$.

A Morse function $f : X \to \mathbb{R}$ is the restriction of a smooth function $\tilde{f} : M \to \mathbb{R}$ such that the following conditions hold :

(1) f is proper and the critical values of f are distinct, i.e if x and y are two distinct critical points of f then $f(x) \neq f(y)$,
(2) for each stratum S of X, the critical points of $f_{|S}$ are nondegenerate,
(3) at each critical point x, the differential $d\tilde{f}(x)$ does not annihilate any generalized tangent space at x.

Let $f : X \to \mathbb{R}$ be a Morse function, S a stratum of X and x a critical point of $f_{|S}$. Let us write $\nu = f(x)$. For sufficiently small $\delta > 0$, let $B(x, \delta)$ be the n-dimensional ball centered at x and of radius δ. The local Morse data for f at x is the pair :

$$\left(B(x,\delta) \cap f^{-1}([\nu - \epsilon, \nu + \epsilon]), B(x,\delta) \cap f^{-1}(\nu - \epsilon)\right),$$

where $0 < \epsilon \ll \delta \ll 1$. This definition is justified by the following property (see [G.M-P] or [Ham]). There exists an open subset A in $\mathbb{R}^+ \times \mathbb{R}^+$ such that

(1) the closure $\overline{A}$ of A in $\mathbb{R}^2$ contains an interval $]0, \delta[$, $\delta > 0$, such that for all $a \in]0, \delta[$, the set $\{b \in \mathbb{R}^+ \mid (a, b) \in A\}$ contains an open interval $]0, \epsilon(a)[$ with $\epsilon(a) > 0$,
(2) for all $(\delta, \epsilon) \in A$, the above pairs of spaces are homeomorphic.

The local Morse data are Morse data in the sense that $f^{-1}(] - \infty, \nu + \epsilon[$ is homeomorphic to the space one gets by attaching $B(x, \delta) \cap f^{-1}([\nu - \epsilon, \nu + \epsilon])$ at $f^{-1}(]-\infty, \nu - \epsilon[$ along $B(x, \delta) \cap f^{-1}(\nu - \epsilon)$ (see [G.M-P, I 3.5]). If x belongs to a stratum $S(x)$ such that $\dim(S(x)) = \dim(X) = d$, the local Morse data at x are homeomorphic to the classical Morse data $(B^\lambda \times B^{d-\lambda}, \partial B^\lambda \times B^{d-\lambda})$ where B^p denotes the p-dimensional unit ball and λ is the Morse index of f at x.

If x belongs to a zero-dimensional stratum then $B(x,\delta)\cap f^{-1}([\nu-\epsilon,\nu+\epsilon])$ has the structure of a cone (see [G.M-P], 3.11). If x lies in a stratum $S(x)$ with $0<\dim(S(x))<\dim(X)$, then one can consider the classical Morse data of $f_{|S(x)}$ at x. We will call them tangential Morse data and denote them by (P_{tg},Q_{tg}). One may choose a normal slice of V at x, that is a closed submanifold of M of dimension $n-\dim(S(x))$, which intersects $S(x)$ in x orthogonally. One defines the normal Morse data (P_{nor},Q_{nor}) at x to be the local Morse data of $f_{|X\cap V}$ at x. Goresky and Mac-Pherson proved that :

- the normal Morse data are well defined, that is to say they are independent of the Riemannian metric and the choice of the normal slice ([G.M-P, I 3.6]),
- the local Morse data (P,Q) of f at x are the product of the tangential and the normal Morse data ([G.M-P, I 3.7) :

$$(P,Q)\approxeq(P_{tg}\times P_{nor},P_{tg}\times Q_{nor}\cup P_{nor}\times Q_{tg}).$$

This implies that $P=B(x,\delta)\cap f^{-1}([\nu-\epsilon,\nu+\epsilon])$ has the structure of a cone.

Definition 2.1. *Let $x\in X$ be a critical point of the Morse function $f: X\to\mathbb{R}$. Let (P,Q) be the local Morse data of f at x. The Euler-Poincaré characteristic $\chi(P,Q)=1-\chi(Q)$ is called the index of f at x and is denoted by $\alpha(X,f,x)$. If $x\in X$ is not a critical point of f, we set $\alpha(X,f,x)=0$.*

If (P_{tg},Q_{tg}) and (P_{nor},Q_{nor}) are the tangential and normal Morse data, then one has :

$$\chi(P,Q)=\chi(P_{tg},Q_{tg})\cdot\chi(P_{nor},Q_{nor}).$$

We will write $\alpha(X,f,x)=\alpha_{tg}(X,f,x)\cdot\alpha_{nor}(X,f,x)$, where $\alpha_{tg}(X,f,x)$ is the tangential Morse index and $\alpha_{nor}(X,f,x)$ is the normal Morse index. We will use the following theorem.

Theorem 2.2. *Let $X\subset M$ be a compact Whitney-stratified set and let $f:X\to\mathbb{R}$ be a Morse function. One has*

$$\chi(X)=\sum_{x\in X}\alpha(X,f,x).$$

Let us explain how this theory is applied to the case of manifolds with corners. Before we recall some basic facts about manifolds with corners. Our reference is [Ce]. A manifold with corners N is defined by an atlas of charts modelled on open subsets of $\mathbb{R}^n_+$. We write ∂N for its boundary. We will make the additional assumption that the boundary is partitioned into pieces $\partial_i N$, themselves manifolds with corners, such that in each chart, the intersections with the coordinate hyperplanes $x_j=0$ correspond to distinct pieces $\partial_i N$ of the boundary. For any set I of suffices, we write $\partial_I N=\cap_{i\in I}\partial_i N$ and we make the convention that $\partial_\emptyset N=N\setminus\partial N$.

Any n-manifold N with corners can be embedded in a n-manifold N^+ without boundary so that the pieces $\partial_i N$ extend to submanifolds $\partial_i N^+$ of codimension 1 in N^+. We will assume that N^+ is provided with a Riemannian metric.

Let N be a manifold with corners and $\omega : N^+ \to \mathbb{R}$ a smooth map. We consider the points P which are critical points of $\omega_{|\partial_I N^+}$.

Definition 2.3. *A critical point P is correct (respectively Morse correct) if, taking $I(P) := \{i \mid P \in \partial_i N\}$, P is a critical (respectively Morse critical) point of $\omega_{|\partial_{I(P)} N^+}$, and is not a critical point of $\omega_{|\partial_J N^+}$ for any proper subset J of $I(P)$.*

Note that a 0-dimensional corner point P is always a critical point because in that case $\partial_{I(P)} N^+ = \{P\}$, which is a 0-dimensional manifold.

Definition 2.4. *The maps ω with all critical points Morse correct are called Morse correct.*

Proposition 2.5. *The set of Morse correct functions is dense and open in the space of all maps $N^+ \to \mathbb{R}$.*

Proof. It is clear from classical Morse theory, because there is a finite number of pieces $\partial_I N^+$. □

The index $\lambda(P)$ of ω at a Morse correct point P is defined to be that of $\omega_{|\partial_{I(P)} N^+}$. If P is a correct critical point of ω, $i \in I(P)$, and J is formed from $I(P)$ by deleting i, then in a chart at P with $\partial_J N$ mapping to $\mathbb{R}^p_+$ and $\partial_{I(P)} N$ to the subset $x_1 = 0$, the function ω is non-critical, but its restriction to $x_1 = 0$ is. Hence $\partial \omega / \partial x_1 \neq 0$.

Definition 2.6. *We say that ω is inward at P if, for each $i \in I(P)$, we have $\partial \omega / \partial x_1 > 0$.*

Remark 2.7. *By our convention, if $I(P) = \emptyset$, then ω is inward at P.*

Theorem 2.8. *If N is compact and ω is Morse correct,*

$$\chi(N) = \sum \left\{ (-1)^{\lambda(P)} \mid P \text{ inward Morse critical point} \right\}.$$

Proof. This is a consequence of Theorem 2.2. See [Dut3], Theorem 2.6. □

In [BK], Bröcker and Kuppe considered a class of stratified sets that they called tame stratified sets. Before giving the precise definition of these sets, we need some notations. For every v in S^{n-1}, v^* is the function defined by $v^*(x) = -\langle v, x \rangle$. For every y in $\mathbb{R}^n$, g_y is the function defined by $g_y(x) = \|y - x\|^2$.

Definition 2.9. *A closed Whitney-stratified set $X \subset \mathbb{R}^n$ is called tame, if the following conditions hold :*

(i) *if S is a stratum of X, then the set $\{\lim T_{q_k} S \mid (q_k) \to p \in \partial S\}$ of all generalized tangent spaces coming from S has Hausdorff dimension less than the dimension of S in the affine grassmannian,*

(ii) *there exists $\rho > 0$ such that for almost all $y \in \mathbb{R}^n$, one has :*

$$\int_{\mathbb{R}^n} \sum_{x \in X, \|y-x\| \leq \rho} |\alpha(X, g_y, x)| dy < +\infty.$$

Lemma 2.10. *If X satisfies condition* (i)*, the following statements hold :*

(1) *the set of couples (v, x) such that $x \in X$, $\|v\| = 1$ and such that there exists a generalized tangent space T at x with $v \perp T$, has Haussdorff dimension less than $n-1$ in the tangent bundle $T\mathbb{R}^n$,*
(2) *for almost all v, the function $v^*_{|X} : X \to \mathbb{R}$, $x \mapsto -\langle v, x\rangle$ is a Morse function,*
(3) *for almost all y, the function $g_{y|X} : X \to \mathbb{R}$, $x \mapsto \|y-x\|^2$ is a Morse function.*

The following classes of sets admit tame stratifications (see [BK], p298) : piecewise linear sets, semi-algebraic sets, subanalytic sets, sets which belong to an o-minimal structure, $\mathcal{X}$-sets (see [Sh]), manifolds with boundary, and Riemannian polyhedra.

Bröcker and Kuppe gave a Gauss-Bonnet formula for any compact tame Whitney stratified set $X \subset \mathbb{R}^n$. They defined a curvature measure $\lambda_0(U)$ on Borel sets U of X by :

$$\lambda_0(U) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in U} \alpha(X, v^*, x) dv,$$

and proved the following proposition.

Proposition 2.11. *Let X be a compact tame Whitney-stratified set in $\mathbb{R}^n$. Then*

$$\lambda_0(X) = \chi(X).$$

Then they explained how the measure $\lambda_0(X)$ can be obtained by approximation. For this, they introduced the notion of approximating functions from outside.

Definition 2.12. *Let $X \subset \mathbb{R}^n$ be a compact tame Whitney stratified set. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is an approximating function for X from outside if the following conditions hold :*

(i) *The function f is nonnegative and $f^{-1}\{0\} = X$.*
(ii) *For every open subset U such that $X \subset U$, there exists $\epsilon > 0$ such that $f^{-1}([0, \epsilon]) \subset U$,*
(iii) *The function f is of class C^3 on $\mathbb{R}^n \setminus X$.*
(iv) *There exists $\delta > 0$ such that all $t \in]0, \delta]$ are regular values of f.*
(v) *If $(y_k)_{k \in \mathbb{N}}$ is a sequence of points in $\mathbb{R}^n$ tending to a point x in X such that $f(y_k) \in]0, \delta]$ and $\frac{\nabla f}{\|\nabla f\|}(y_k)$ tends to v, then v is normal to $T_x S(x)$.*

(vi) *For* $t \in [0,\delta]$, *let* X_t *denote* $f^{-1}([0,t])$. *There exist* $\rho > 0$ *and a function* $M : \mathbb{R}^n \to [0,+\infty[$ *with* $\int_{\mathbb{R}^n} M(x)dx < +\infty$ *such that for all* t *with* $0 < t < \delta$ *and for almost all* $y \in \mathbb{R}^n$, *one has :*

$$\sum_{x \in X_t, \|y-x\| \leq \rho} |\alpha(X_t, g_y, x)| \leq M(x).$$

They proved the two following results that will be very useful for us.

Proposition 2.13. *Let* $X \subset \mathbb{R}^n$ *be a compact tame Whitney-stratified set which belongs to an o-minimal system (resp. an* $\mathcal{X}$*-system). Then* X *admits an approximating function from outside which belongs to the same o-minimal system (resp.* $\mathcal{X}$*-system).*

Proof. See [BK], Proposition 7.1. □

Theorem 2.14. *Let* f *be an approximating function for* X *from outside. Let* $U \subset \mathbb{R}^n$ *be an open set such that the bad set*

$$B(U) = \{v \in \mathbb{R}^n \mid \exists x \in \mathrm{Bd}(U) \cap X \text{ with } v \perp T_x S(x)\}$$

has measure 0 *in* $\mathbb{R}^n$. *Then for* $t \to 0$, *one has*

$$\lambda_0(X_t \cap U) \to \lambda_0(X \cap U).$$

Proof. See [BK], Theorem 6.2. □

This theorem can be applied for example if U is an euclidian ball whose boundary intersects the strata of X tranversally.

3. Study of critical points

Let X be a closed semi-algebraic set in $\mathbb{R}^n$. It admits a finite Whitney stratification, each stratum being semi-algebraic.

For every $R > 0$, let $f_R : \mathbb{R}^n \to \mathbb{R}$ be a semi-algebraic approximating function from outside for $X \cap B^n_{2R}$.

Lemma 3.1. *For every* $R > 0$, *there exists* δ_R *with* $0 < \delta_R \ll 1$ *such that* S^{n-1}_R *intersects* $f_R^{-1}(t)$ *transversally for every* $t \in]0, \delta_R]$.

Proof. The function $\tilde{f_R} : S^{n-1}_R \setminus X \to \mathbb{R}$, $x \mapsto f_R(x)$ is semi-algebraic of class C^3, hence it admits a finite number of critical values. We choose δ_R smaller than any critical value of $\tilde{f_R}$. □

Let $R > 0$ be sufficiently big so that S^{n-1}_R intersects each stratum transversally. We are going to study the critical points of the functions v^* on $X \cap S^{n-1}_R$ and on $f_R^{-1}(t) \cap S^{n-1}_R$ for $t \in]0, \delta_R]$.

For every $v \in S^{n-1}$, let $\Gamma_{R,v}$ be the following polar set :

$$\Gamma_{R,v} = \left\{ x \in S^{n-1}_R \setminus X \mid \mathrm{rank}(v, \nabla f_R(x), x) < 3 \right\}.$$

Lemma 3.2. *For almost all* $v \in S^{n-1}$, $\Gamma_{R,v} \cap f_R^{-1}(]0, \delta_R])$ *is a semi-algebraic curve of class* C^3.

Proof. Let

$$Y = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \forall (i,j,k) \in \{1,\ldots,n\}^3 \ m_{ijk}(x,y) = 0\right\},$$

where

$$m_{ijk}(x,y) = \begin{vmatrix} x_i & x_j & x_k \\ \frac{\partial f_R}{\partial x_i}(x) & \frac{\partial f_R}{\partial x_j}(x) & \frac{\partial f_R}{\partial x_k}(x) \\ y_i & y_j & y_k \end{vmatrix}.$$

The set

$$Y' = Y \cap \left(f_R^{-1}(]0,\delta_R]) \times (\mathbb{R}^n \setminus \{(0,\ldots,0)\})\right)$$

is a semi-algebraic manifold of dimension C^3 of dimension $n+1$. To see this, take a point (x,y) in Y'. We can assume that

$$\begin{vmatrix} x_1 & x_2 \\ \frac{\partial f_R}{\partial x_1}(x) & \frac{\partial f_R}{\partial x_2}(x) \end{vmatrix} \neq 0.$$

This implies that in a neighborhood of (x,y), Y' is defined by the vanishing of the functions $m_{123},\ldots,m_{12n}$ and $x_1^2+\cdots+x_n^2-R$ because for every $(i,j,k) \in \{1,\ldots,n\}^3$, we have :

$$\begin{vmatrix} x_1 & x_2 \\ \frac{\partial f_R}{\partial x_1}(x) & \frac{\partial f_R}{\partial x_2}(x) \end{vmatrix} \cdot m_{ijk}(x,y) = \begin{vmatrix} x_j & x_k \\ \frac{\partial f_R}{\partial x_j}(x) & \frac{\partial f_R}{\partial x_k}(x) \end{vmatrix} \cdot m_{12i}(x,y) -$$

$$\begin{vmatrix} x_i & x_k \\ \frac{\partial f_R}{\partial x_i}(x) & \frac{\partial f_R}{\partial x_k}(x) \end{vmatrix} \cdot m_{12j}(x,y) + \begin{vmatrix} x_i & x_j \\ \frac{\partial f_R}{\partial x_i}(x) & \frac{\partial f_R}{\partial x_j}(x) \end{vmatrix} \cdot m_{12k}(x,y).$$

A simple computation of determinants shows that the gradient vectors of these functions are linearly independent. Let us consider the projection π_y defined as follows :

$$\begin{array}{cccc} \pi_y & : & Y' & \to & \mathbb{R}^n \\ & & (x,y) & \mapsto & y. \end{array}$$

Let Σ_{π_y} be the set of critical values of π_y. We take v in $S^{n-1} \setminus \Sigma_{\pi_y}$. □

Lemma 3.3. *For almost all $v \in S^{n-1}$, there exists $\delta_v > 0$ such that for every t with $0 < t \leq \delta_v$, $v^*_{|f_R^{-1}(t) \cap S_R^{n-1}}$ is a Morse function.*

Proof. Let v be in $S^{n-1} \setminus \Sigma_{\pi_y}$ where Σ_{π_y} is defined in the previous lemma. Let L be a connected component of $\Gamma_{R,v}$ which contains a point x of X in its closure. In a neighborhood of x, L can be parametrized by an analytic arc $\gamma : [0,\nu] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(]0,\nu]) \subset L$. We have $f_R(\gamma(0)) = 0$ and $f_R(\gamma(s)) > 0$ for all $s \in]0,\nu[$. There exists ν' such that $(f_R \circ \gamma)'$ does not vanish on $]0,\nu'[$. In fact $(f_R \circ \gamma)'$ is a C^2 semi-algebraic function on $]0,\nu[$. If $(f \circ \gamma)' \equiv 0$ on $]0,\nu[$ then $f \circ \gamma$ is constant on $]0,\nu[$ and, since it is continuous on $[0,\nu[$, it must be equal to 0, which is impossible. Hence $(f_R \circ \gamma)'$ admits a finite number of zeroes in $]0,\nu[$. We take ν' to be the smallest of these zeroes.

Furthermore one can assume that $\begin{vmatrix} x_1 & x_2 \\ \frac{\partial f_R}{\partial x_1} & \frac{\partial f_R}{\partial x_2} \end{vmatrix}$ does not vanish on $]0,\nu'[$, changing ν' by a smaller positive number if necessary. In that case, $\gamma(]0,\nu'[)$ is defined by the vanishing of the functions $m_{123}^v,\ldots,m_{12n}^v$ and $\omega_R = x_1^2 + \cdots + x_n^2 - R$, where $m_{12i}^v(x) = m_{12i}(x,v)$. One has that for all $s \in]0,\nu'[$ $\langle \nabla f_R(\gamma(s)), \gamma'(s)\rangle = (f_R \circ \gamma)'(s) \neq 0$. Hence the gradient vectors $\nabla f_R(\gamma(s))$, $\nabla w_R(\gamma(s))$, $\nabla m_{123}^v(\gamma(s)),\ldots,\nabla m_{12n}^v(\gamma(s))$ are linearly independent. By Lemma 3.2 in [Sz], this implies that the function

$$v^* : f_R^{-1}(f_R(\gamma(s))) \cap S_R^{n-1} \to \mathbb{R}$$

has a Morse critical point at $\gamma(s)$. One takes for δ_v the minimum, taken over all the arcs of $\Gamma_{R,v}$ containing a point of X in their closure, of the ν''s. $\square$

Let v be a vector in S^{n-1} which satisfies the properties of the two previous lemmas. Let us write the decomposition of $\Gamma_{R,v}$ into the union of its connected components :

$$\Gamma_{R,v} = L_1 \sqcup \ldots \sqcup L_{m_{R,v}} \sqcup M_1 \sqcup \ldots \sqcup M_{p_{R,v}},$$

where for $i \in \{1,\ldots,m_{R,v}\}$, $\overline{L_i} \cap X \neq \emptyset$ and for $i \in \{1,\ldots,p_{R,v}\}$, $\overline{M_i} \cap X = \emptyset$. For $t \in]0,\delta_v[$ and for $i \in \{1,\ldots,m_{R,v}\}$, let $q_i^{t,R,v}$ be the intersection point of L_i and $f_R^{-1}(t)$. This point is a Morse critical point of $v*_{|S_R^{n-1} \cap f_R^{-1}(t)}$. For $i \in \{1,\ldots,m_{R,v}\}$, let $q_i^{R,v}$ be the intersection point of $\overline{L_i}$ and X.

Lemma 3.4. *The points $q_i^{R,v}$ are critical points of $v^*_{|X \cap S_R^{n-1}}$.*

Proof. Let us fix $i \in \{1,\ldots,m_{R,v}\}$. For $t \in]0,\delta_v[$, there exist two real numbers λ_t and μ_t such that

$$v = \lambda_t \cdot \frac{\nabla f_R}{\|\nabla f_R\|}(q_i^{t,R,v}) + \mu_t \cdot \frac{q_i^{t,R,v}}{\|q_i^{t,R,v}\|}.$$

Applying this equality to $t = \frac{1}{k}$, we obtain a sequence of points $p_k := q_i^{\frac{1}{k},R,v}$ in S_R^{n-1} and two sequences of real numbers $\alpha_k := \lambda_{\frac{1}{k}}$ and $\beta_k := \mu_{\frac{1}{k}}$ such that :

$$v = \alpha_k \cdot \frac{\nabla f_R}{\|\nabla f_R\|}(p_k) + \beta_k \cdot \frac{p_k}{\|p_k\|}. \tag{1}$$

Taking a subsequence if necessary, one can assume that $\frac{\nabla f_R}{\|\nabla f_R\|}(p_k)$ tends to a vector w in S^{n-1}. By condition (v) in Definition 2.12, w is perpendicular to $T_{q_i^{R,v}} S(q_i^{R,v})$. Moreover, since S_R^{n-1} intersects the strata of X transversally, w and $\frac{q_i^{R,v}}{\|q_i^{R,v}\|}$ are not colinear and $|\langle w, \frac{q_i^{R,v}}{\|q_i^{R,v}\|}\rangle| < 1$. Hence there exist C with $0 \leq C < 1$ and k_0 such that for every $k \geq k_0$, one has

$$\left| \langle \frac{\nabla f_R}{\|\nabla f_R\|}(p_k), \frac{p_k}{\|p_k\|}\rangle \right| \leq C.$$

Since $\langle v,v\rangle = 1$, this implies that for $k \geq k_0$, $\alpha_k^2 + \beta_k^2 + 2C\alpha_k\beta_k \leq 1$ or $\alpha_k^2 + \beta_k^2 - 2C\alpha_k\beta_k \leq 1$. Then it is not difficult to see that $(\alpha_k)_{k\geq k_0}$ and $(\beta_k)_{k\geq k_0}$ are bounded. Taking subsequences if necessary, one can assume that α_k tends to a real number α and that β_k tends to a real β. Taking the limit in equality (1), we obtain :

$$v = \alpha \cdot w + \beta \cdot \frac{q_i^{R,v}}{\|q_i^{R,v}\|}.$$

This means exactly that $q_i^{R,v}$ is a critical point of $v^*_{|X\cap S_R^{n-1}}$. □

We are going to study the critical points of $v^*_{|X\cap S_R^{n-1}}$ for v generic and R sufficiently big. Let $\Sigma(X)$ be the subset of S^{n-1} defined as follows. A vector v belongs to $\Sigma(X)$ if there exists a sequence $(x_k)_{k\in\mathbb{N}}$ such that $\|x_k\| \to +\infty$ and a sequence $(v_k)_{k\in\mathbb{N}}$ of vectors in S^{n-1} such that $v_k \perp T_{x_k}S(x_k)$ and $v_k \to v$. We want to prove that $\Sigma(X)$ is a semi-algebraic set of dimension less than $n-1$.

Lemma 3.5. *Let X be a closed semi-algebraic set in $\mathbb{R}^n$. The set $\Sigma(X)$ is a semi-algebraic set of S^{n-1} of dimension less than $n-1$.*

Proof. If we write $X = \cup_{i=1}^r S_i$, where $(S_i)_{1\leq i\leq r}$ is a finite semi-algebraic Whitney stratification of X, then we see that $\Sigma(X) = \cup_{i=1}^r \Sigma(S_i)$. Hence it is enough to prove the lemma when X is a smooth semi-algebraic manifold of dimension $n-k$, $0<k<n$.

Let us take $x = (x_1,\ldots,x_n)$ as a coordinate system for $\mathbb{R}^n$ and (x_0,x) for $\mathbb{R}^{n+1}$. Let φ be the semi-algebraic diffeomorphism between $\mathbb{R}^n$ and $S^n \cap \{x_0>0\}$ given by

$$\varphi(x) = \left(\frac{1}{\sqrt{1+\|x\|^2}}, \frac{x_1}{\sqrt{1+\|x\|^2}}, \ldots, \frac{x_n}{\sqrt{1+\|x\|^2}}\right).$$

Observe that $(x_0,x) = \varphi(z)$ if and only if $z = \frac{x}{x_0}$. The set $\varphi(X)$ is a smooth semi-algebraic set of dimension $n-k$. Let M be the following semi-algebraic set :

$$M = \left\{(x_0,x,y) \in \mathbb{R}^{n+1}\times\mathbb{R}^n \mid (x_0,x)\in\varphi(X) \text{ and } y \perp T_{\frac{x}{x_0}}X\right\}.$$

We will show that M is a smooth manifold of dimension n. Let $p=(x_0,x,y)$ be a point in M and let $z = \varphi^{-1}(x_0,x) = \frac{x}{x_0}$. In a neighborhood of z, X is defined by the vanishing of smooth functions $g_1,\ldots,g_k$. For $i\in\{1,\ldots,k\}$, let G_i be the smooth function defined by

$$G_i(x_0,x) = g_i\left(\frac{x}{x_0}\right) = g_i(\varphi^{-1}(x_0,x)).$$

Then in a neighborhood of (x_0,x), $\varphi(X)$ is defined by the vanishing of $G_1,\ldots,G_k$ and $x_0^2+x_1^2+\cdots+x_n^2-1$. Note that for $i,k\in\{1,\ldots,n\}^2$, $\frac{\partial G_i}{\partial x_k}(x_0,x) = \frac{1}{x_0}\cdot\frac{\partial g_i}{\partial x_k}(x)$. Hence in a neighborhood of p, M is defined by the vanishing of $G_1, \ldots, G_k$, $x_0^2 + x_1^2 + \cdots + x_n^2 - 1$ and the following minors $m_{i_1 i_2 \ldots i_{k+1}}$, $(i_1, \ldots, i_{k+1}) \in \{1, \ldots, n\}^{k+1}$, given by :

$$m_{i_1 i_2 \ldots i_{k+1}}(x_0, x, y) = \begin{vmatrix} \frac{\partial G_1}{\partial x_{i_1}}(x_0,x) & \cdots & \frac{\partial G_1}{\partial x_{i_{k+1}}}(x_0,x) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_k}{\partial x_{i_1}}(x_0,x) & \cdots & \frac{\partial G_k}{\partial x_{i_{k+1}}}(x_0,x) \\ y_{i_1} & \cdots & y_{i_{k+1}} \end{vmatrix}.$$

Since $\operatorname{rank}(\nabla g_1, \ldots, \nabla g_k) = k$ at $z = \varphi^{-1}(x_0, x)$, one can assume that

$$\begin{vmatrix} \frac{\partial G_1}{\partial x_1}(x_0,x) & \cdots & \frac{\partial G_1}{\partial x_k}(x_0,x) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_k}{\partial x_1}(x_0,x) & \cdots & \frac{\partial G_k}{\partial x_k}(x_0,x) \end{vmatrix} \neq 0.$$

This implies that around p, M is defined by the vanishing of $G_1, \ldots, G_k$, $m_{1\ldots k k+1}, \ldots, m_{1\ldots kn}$ and $x_0^2 + x_1^2 + \cdots + x_n^2 - 1$ (a similar argument is given and proved in [Dut2], p316-318). It is straightforward to see that the gradient vectors of these functions are linearly independent. Then $\bar{M} \setminus M$ is a semi-algebraic set of dimension less than n. If $\pi_y : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection on the last n coordinates, then we have $\Sigma(X) = S^{n-1} \cap \pi_y(\bar{M} \setminus M)$. $\square$

Corollary 3.6. *If v does not belong to $\Sigma(X)$, then the set of critical points of $v^*_{|X}$ is compact.*

Proof. If this set of critical points is not compact, then there exists a sequence of points $(x_k)_{k \in \mathbb{N}}$ in X such that $\|x_k\| \to +\infty$ and $v \perp T_{x_k} S(x_k)$. $\square$

Lemma 3.7. *There exists a semi-algebraic set $\Sigma'(X) \subset S^{n-1}$ of dimension less than $n-1$ such that if v does not belong to $\Sigma'(X)$, then $X \cap \{x \in \mathbb{R}^n \mid \operatorname{rank}(x, v) < 2\}$ is finite.*

Proof. Since the stratification of X is finite and semi-algebraic, it is enough to prove the lemma for a smooth semi-algebraic stratum S of dimension s. Let N_S be the following semi-algebraic set :

$$N_S = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in S \text{ and } \forall (i,j) \in \{1, \ldots, n\}^2, \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = 0 \right\}.$$

Using the same kind of arguments as in the previous lemmas, we see that the $N_S \setminus \{\{0\} \times \mathbb{R}^n\}$ is a smooth manifold of dimension $s+1$. Let

$$\begin{array}{cccc} \pi_y & : & N_S & \to & \mathbb{R}^n \\ & & (x,y) & \mapsto & y \end{array}$$

be the projection onto the n last coordinates. The Bertini-Sard theorem (see [BCR], Proposition 2.8.12) implies that the set Σ_{π_y} of critical values of π_y is semi-algebraic of dimension less than n. One takes $\Sigma'(X) = S^{n-1} \cap \Sigma_{\pi_y}$.

Observe that $X \cap \{x \in \mathbb{R}^n \mid \mathrm{rank}(x,v) < 2\} = \pi_y^{-1}\{v\}$ if $0 \notin X$ and $X \cap \{x \in \mathbb{R}^n \mid \mathrm{rank}(x,v) < 2\} = \pi_y^{-1}\{v\} \cup \{0\}$ if $0 \in X$. $\square$

Corollary 3.8. *Let v be vector in S^{n-1}. If there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of points in X such that*

- $\|x_k\| \to +\infty$,
- $v \in N_{x_k}S(x_k) \oplus \mathbb{R} \cdot x_k$,
- $\lim_{k\to+\infty} |v^*(x_k)| < +\infty$,

then v belongs to $\Sigma(X)$ (Here $N_{x_k}S(x_k)$ is the normal space to the stratum $S(x_k)$).

Proof. We can assume that $v = -e_1 = (-1, 0, \ldots, 0)$. In this case, $v^* = x_1$. Furthermore, since the stratification is finite, one can assume that $(x_k)_{k\in\mathbb{N}}$ is a sequence of points lying in a stratum S. By the version at infinity of the Curve Selection Lemma (See [NZ], Lemma 2), there exists an analytic curve $p(t) :]0, \varepsilon[\to S$ such that $\lim_{t\to 0} \|p(t)\| = +\infty$, $\lim_{t\to 0} p_1(t) < +\infty$ and such that for $t \in]0, \varepsilon[$, $-e_1$ belongs to the space $N_{p(t)}S \oplus \mathbb{R} \cdot p(t)$. Let us consider the expansions as Laurent series of the p_i's :

$$p_i(t) = a_i t^{\alpha_i} + \cdots,\ i = 1, \ldots, n.$$

Let α be the minimum of the α_i's. Necessarily, $\alpha < 0$ and $\alpha_1 \geq 0$. It is straightforward to see that $\|p(t)\|$ has an expansion of the form :

$$\|p(t)\| = bt^{\alpha} + \cdots,\ b > 0.$$

Let us denote by π_t the orthogonal projection onto $T_{p(t)}S$. For every $t \in]0, \varepsilon[$, there exists a real number $\lambda(t)$ such that :

$$\pi_t(e_1) = \lambda(t).\pi_t(p(t)) = \lambda(t) \cdot \|\pi_t(p(t))\| \cdot \frac{\pi_t(p(t))}{\|\pi_t(p(t))\|}.$$

Observe that taking ε sufficiently small, we can assume that $\pi_t(p(t))$ does not vanish, for $S^{n-1}_{\|p(t)\|}$ intersects S transversally. Using the fact that $p'(t)$ is tangent to S at $p(t)$, we find that :

$$p_1'(t) = \langle p'(t), e_1 \rangle = \langle p'(t), \pi_t(e_1) \rangle = \lambda(t) \cdot \langle p'(t), p(t) \rangle.$$

This implies that $\mathrm{ord}(\lambda) \geq \alpha_1 - 2\alpha$. Let β be the order of $\|\pi_t(p)\|$. Since $\|p(t)\| \geq \|\pi_t(p(t))\|$, β is greater or equal to α. Finally we obtain that $\mathrm{ord}(\lambda \|\pi_t(p(t))\|)$ is greater or equal to $\alpha_1 - 2\alpha + \beta$, which is positive. This proves the corollary. $\square$

We recall that we have chosen $R > 0$ sufficiently big so that S^{n-1}_R intersects every stratum of X transversally. Let us consider a function $f : X \to \mathbb{R}$. Let $q \in X \cap S^{n-1}_R$ be a critical point of $f_{|S(q)\cap S^{n-1}_R}$ We will say that $f_{|X\cap B^n_R}$ is inward (resp. outward) at q of if $f_{|S(q)\cap B^n_R}$ is inward (resp. outward) at q.

Corollary 3.9. *Let v be a vector in S^{n-1} not belonging to $\Sigma(X) \cup \Sigma'(X)$. There exits $R_v > 0$ such that for every $R \geq R_v$, $v^*_{|X\cap B^n_R}$ is not inward* *(resp. outward) at the critical points of* $v^*_{|X\cap S_R^{n-1}}$ *lying in* $\{v^*>0\}$ *(resp in* $\{v^*<0\}$*).*

Proof. By Corollary 3.6, we know that if v does not belong to $\Sigma(X)$ then $\{v^*=0\}$ intersects each stratum transversally outside a compact set. Let us remark also that for R sufficiently big and for every stratum S of X, S_R^{n-1} intersects $\{v^*=0\}\cap S$ transversally. Hence the critical points of $v^*_{|X\cap S_R^{n-1}}$ do not lie on the level $\{v^*=0\}$ if R is big enough.

Let us fix a stratum S of dimension s and a vector v in S^{n-1} not belonging to $\Sigma(X)\cup\Sigma'(X)$. By Corollary 3.6 and Lemma 3.7, the critical points of $v^*_{|S\cap S_R^{n-1}}$ are correct if R is sufficiently big. We can assume that $v=-e_1$. Let us suppose that there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $S\cap\{v^*>0\}$ such that $\|x_k\|\to+\infty$, x_k is a critical point of $v^*_{|S\cap S^{n-1}_{\|x_k\|}}$ and $v^*_{|S\cap B^n_{\|x_k\|}}$ is inward at x_k. By the Curve Selection Lemma at infinity, there exists an analytic curve $p(t):]0,\varepsilon[\to S$ such that $\lim_{t\to0}\|p(t)\|=+\infty$ and $p(t)$ is a critical point of $v^*_{|S\cap S^{n-1}_{\|p(t)\|}}$ and $v^*_{|S\cap B^n_{\|p(t)\|}}$ is inward at $p(t)$. Keeping the notations of Corollary 3.8, we have that for every $t\in]0,\varepsilon[$, there exists a real number $\lambda(t)>0$ such that :

$$\pi_t(e_1)=\lambda(t).\pi_t(p(t))=\lambda(t)\cdot\|\pi_t(p(t))\|\cdot\frac{\pi_t(p(t))}{\|\pi_t(p(t))\|}.$$

But we know that $p_1'(t)=\lambda(t)\langle p'(t),p(t)\rangle$, which can be written $p_1'(t)=\frac{1}{2}\lambda(t)(\|p(t)\|^2)'$. Hence the function p_1 is increasing and thus the function $v^*\circ p$ is decreasing. Since it is bounded from above as t tends to 0, we find that $\lim_{t\to0}|v^*(p(t))|<+\infty$, a contradiction. $\square$

Corollary 3.10. *Let v be a vector in S^{n-1} not belonging to $\Sigma(X)\cup\Sigma'(X)$. There exits $R_v>0$ such that for every $R\geq R_v$, there exists $\tau_{R,v}$ such that for every t with $0<t\leq\tau_{R,v}$, $v^*_{|f_R^{-1}(t)\cap B_R^n}$ is not inward (resp. outward) at the critical points of $v^*_{|f_R^{-1}(t)\cap S_R^{n-1}}$ lying in $\{v^*>0\}$ (resp. in $\{v^*<0\}$).*

Proof. Let us choose v not in $\Sigma(X)\cup\Sigma'(X)$ and R_v defined in the previous corollary. Let R be greater or equal to R_v. Since $f_{R|\{v^*=0\}\cap S_R^{n-1}}$ is a semi-algebraic function of class C^3 outside X, it has a finite number of critical values. Hence for t positive and sufficiently small, the critical points of $v^*_{|f_R^{-1}(t)\cap S_R^{n-1}}$ do not lie on the level $\{v^*=0\}$.

Let us assume that for every $R'\geq R_v$, there exists $R\geq R'$ such that for every $\tau>0$ there exists $t\leq\tau$ such that $v^*_{|f_R^{-1}(t)\cap B_R^n}$ is inward at a critical point of $v^*_{|f_R^{-1}(t)\cap S_R^{n-1}}$ lying in $\{v^*>0\}$. One can construct a sequence $(R_l)_{l\in\mathbb{N}}$ of positive numbers such that $\lim_{l\to+\infty}R_l=+\infty$ and, for each $l\in\mathbb{N}$, a sequence of points $(y_k^l)_{k\in\mathbb{N}}$ in $S_{R_l}^{n-1}$ such that y_k^l is a critical point of $v^*_{|f_{R_l}^{-1}(t_k)\cap S_{R_l}^{n-1}}$ at which $v^*_{|f_{R_l}^{-1}(t_k)\cap B_{R_l}^n}$ is inward, where $t_k=f_{R_l}(y_k^l)$

and $\lim_{k\to+\infty} t_k = 0$, and such that $v^*(y_k^l) > 0$. Taking a subsequence if necessary, one can assume that (y_k^l) tends to a point y^l which lies in $X \cap S_{R_l}^{n-1}$. By condition (v) in Definition 2.12, we know that there exists a vector u^l in S^{n-1} normal to $S(y^l)$ such that

$$\lim_{k\to+\infty} \frac{\nabla f_{R_l}(y_k^l)}{\|\nabla f_{R_l}(y_k^l)\|} = u^l.$$

At each point y_k^l, we have the following decomposition of v :

$$v = \mu_k \cdot \frac{\nabla f_{R_l}(y_k^l)}{\|\nabla f_{R_l}(y_k^l)\|} + \lambda_k \cdot \frac{y_k^l}{\|y_k^l\|},$$

where $\lambda_k > 0$. As in Lemma 3.4, it is not difficult to prove that λ_k and μ_k are bounded. Taking subsequences if necessary and taking the limit in the above equality, we find that there exist $\lambda \geq 0$ and μ such that

$$v = \mu.u^l + \lambda.y^l.$$

If $\lambda = 0$ then v and u^l are colinear and y^l is a critical point of $v^*_{|X}$ which is excluded if R_l is big enough. Hence $\lambda > 0$. Furthermore $v^*(y^l) > 0$ for $S_{R_l}^{n-1}$ intersects $X \cap \{v^* = 0\}$ transversally if R_l is big enough. So we have constructed a sequence of points $(y^l)_{l\in\mathbb{N}}$ such that y^l is a critical point of $v^*_{|X\cap S_{R_l}^{n-1}}$ at which $v^*_{|X\cap B_{R_l}^n}$ is inward, and such that $v^*(y^l) > 0$. This is impossible by Corollary 3.9. $\square$

4. The Gauss-Bonnet formula

In this section, we state and prove a Gauss-Bonnet formula for closed semi-algebraic sets. So let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set. Let $(K_R)_{R>0}$ be an exhaustive sequence of compact borelian sets of X, that is a sequence $(K_R)_{R>0}$ of compact borelian sets of X such that $\cup_{R>0} K_R = X$ and $K_R \subseteqq K_{R'}$ if $R \leq R'$. For every $R > 0$, one has :

$$\lambda_0(X \cap K_R) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x\in X\cap K_R} \alpha(X, v^*, x)dv.$$

Moreover the following limit

$$\lim_{R\to+\infty} \sum_{x\in X\cap K_R} \alpha(X, v^*, x)$$

is equal to $\sum_{x\in X} \alpha(X, v^*, x)$ which is uniformly bounded by Hardt's theorem ([Har]). Applying Lebesgue's theorem, we obtain :

$$\lim_{R\to+\infty} \lambda_0(X \cap K_R) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \lim_{R\to+\infty} \sum_{x\in X\cap K_R} \alpha(X, v^*, x)dv =$$

$$\frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x\in X} \alpha(X, v^*, x).$$

Definition 4.1. *One sets*

$$\lambda_0(X) = \lim_{R \to +\infty} \lambda_0(X \cap K_R),$$

where $(K_R)_{R>0}$ is an exhaustive sequence of compact borelian sets of X.

Since this definition does not depend on the choice of the exhaustive sequence, we will study $\lim_{R\to+\infty} \lambda_0(X \cap B_R^n)$. Let $R \gg 1$ be such that S_R^{n-1} intersects all the strata of X transversally. Using the technics of Section 3, it is not difficult to see that the set

$$\{v \in S^{n-1} \mid \exists x \in X \cap S_R^{n-1} \text{ with } v \perp T_x S(x)\}$$

is a semi-algebraic set of dimension less than $n-1$. This implies that :

$$\lambda_0(X \cap B_R^n) = \lambda_0(X \cap \mathring{B}_R^n),$$

where $\mathring{B}_R^n$ is the interior of B_R^n. From now on, we will study $\lambda_0(X \cap \mathring{B}_R^n)$ with $R > 0$ sufficiently big.

Let f_R be a semi-algebraic approximating function from outside for $X \cap B_{2R}^n$. By Theorem 2.14, one has :

$$\lambda_0(X \cap \mathring{B}_R^n) = \lim_{t\to 0} \lambda_0(\{f_R \leq t\} \cap \mathring{B}_R^n).$$

and

$$\lambda_0(\{f_R \leq t\} \cap \mathring{B}_R^n) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in \{f_R \leq t\} \cap \mathring{B}_R} \alpha(\{f_R \leq t\}, v^*, x) dv.$$

Let us evaluate first $\sum_{x \in \{f_R \leq t\} \cap \mathring{B}_R} \alpha(\{f_R \leq t\}, v^*, x)$. The set $\{f_R \leq t\}$ is a C^3 manifold with boundary. Applying stratified Morse Theory to the case of manifolds with boundary, we find that the points x of $\{f_R \leq t\}$ for which the index $\alpha(\{f_R \leq t\}, v^*, x)$ does not vanish are exactly the critical points x of $v^*_{|\{f_R = t\}}$ such that $\nabla v^*(x)$ is a negative multiple of $\nabla f_R(x)$, or equivalently v is a positive multiple of $\nabla f_R(x)$. By the results of Section 3, we known that for R sufficiently big and t sufficiently small, the set $\{f_R \leq t\} \cap B_R^n$ is a manifold with corners. Furthermore, for almost all $v \in S^{n-1}$, the function $v^*_{|\{f_R = t\} \cap S_R^{n-1}}$ is a Morse function (see Lemma 3.3). The function $v^*_{|\{f_R \leq t\} \cap B_R^n}$ is inward at a critical point p of $v^*_{|f_R^{-1}(t) \cap S_R^{n-1}}$ if v, which is also $-\nabla v^*(p)$, admits the following decomposition :

$$v = \lambda \cdot p + \mu \cdot \nabla f_R(p),$$

where $\lambda > 0$ and $\mu > 0$. Let $\{p_j^{t,R,v},\ j = 1, \ldots, l_v^{t,R}\}$ be the set of these critical points and let $\{\sigma_j^{t,R,v},\ i = 1, \ldots, l_v^{t,R}\}$ be the set of their respective

Morse indices. We have :

$$\chi(\{f_R \le t\} \cap B_R^n) = \sum_{x \in \{f_R \le t\} \cap \mathring{B}_R^n} \alpha(\{f_R \le t\}, v^*, x) + \sum_{j=1}^{l_v^{t,R}} (-1)^{\sigma_j^{t,R,v}} + I_v^{t,R},$$

where $I_v^{t,R}$ belongs to $\{-1,0,1\}$ and represents the contribution of the possible critical point of $v^*_{|\{f_R<t\}\cap S_R^{n-1}}$ at which $v^*_{|\{f_R \le t\}\cap B_R^n}$ is inward, or equivalently, at which v^* is negative. This gives :

$$\mathrm{Vol}(S^{n-1}) \cdot \lambda_0(\{f_R \le t\} \cap B_R^n) = \int_{S^{n-1}} \chi(\{f_R \le t\} \cap B_R^n) dv - \int_{S^{n-1}} \left[\sum_{j=1}^{l_v^{t,R}} (-1)^{\sigma_j^{t,R,v}} + I_v^{t,R} \right] dv.$$

Taking the limits as t tends to 0 and R tends to ∞, we obtain :

$$\mathrm{Vol}(S^{n-1}) \cdot \lambda_0(X) = \lim_{R \to +\infty} \lim_{t \to 0} \mathrm{Vol}(S^{n-1}) \cdot \chi(\{f_R \le t\} \cap B_R^n) - \lim_{R \to +\infty} \lim_{t \to 0} \int_{S^{n-1}} \left[\sum_{j=1}^{l_v^{t,R}} (-1)^{\sigma_j^{t,R,v}} + I_v^{t,R} \right] dv.$$

We need to compute the two double limits in the above equality.

Lemma 4.2. *Let $R > 0$ be such that S_R^{n-1} intersects the strata of X transversally. Let f_R be a semi-algebraic approximating from outside for $X \cap B_{2R}^n$. There exists a neighborhood U of $X \cap S_R^{n-1}$ such that for $x \in U \setminus X$, $\nabla f_R(x)$ and x are not colinear.*

Proof. If it is not the case, one can find a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $f_R(x_k)$ tends to 0, $\|x_k\|$ tends to R and $\nabla f_R(x_k)$ is colinear to x_k. Taking a subsequence if necessary, one can assume that (x_k) tends to a point $x \in X \cap S_R^{n-1}$. This implies that the sequence $(\frac{\nabla f_R(x_k)}{\|\nabla f_R(x_k)\|})$ tends to $\pm \frac{x}{\|x\|}$. By condition (v) in Definition 2.12, one can conclude that $x \perp T_x S(x)$, which contradicts the fact that S_R^{n-1} intersects $S(x)$ transversally at x. $\square$

Proposition 4.3. *We have :*

$$\lim_{R \to +\infty} \lim_{t \to 0} \chi(\{f_R \le t\} \cap B_R^n) = \chi(X).$$

Proof. Let R be sufficiently big so that S_R^{n-1} intersects the strata of X transversally. There exits a neighborhood U of $X \cap S_R^{n-1}$ such that for $x \in U \setminus X$, $\nabla f_R(x)$ and x are not colinear. In this situation, one can apply the remark after Theorem 3.2 in [Dut4,p10] to prove that for t sufficiently small $X \cap B_R^n$ is a strong deformation retract of $\{f_R \le t\} \cap B_R^n$. Hence we find that

$$\lim_{t \to 0} \chi(\{f_R \le t\} \cap B_R^n) = \chi(X \cap B_R^n)$$

and thus

$$\lim_{R\to+\infty}\lim_{t\to 0}\chi(\{f_R\le t\}\cap B_R^n)=\chi(X).$$

$\square$

Let us now focus on the second double limit.

Lemma 4.4. *We have :*

$$\lim_{R\to+\infty}\lim_{t\to 0}\int_{S^{n-1}}\left[\sum_{j=1}^{l_v^{t,R}}(-1)^{\sigma_j^{t,R,v}}+I_v^{t,R}\right]dv=$$

$$\int_{S^{n-1}}\lim_{R\to+\infty}\lim_{t\to 0}\left[\sum_{j=1}^{l_v^{t,R}}(-1)^{\sigma_j^{t,R,v}}+I_v^{t,R}\right]dv.$$

Proof. Using the study of the critical points of $v^*_{|f_R^{-1}(t)\cap S_R^{n-1}}$ that we have done in Section 3, we see that for almost all $v\in S^{n-1}$

$$\left|\lim_{t\to 0}\sum_{j=1}^{l_v^{t,R}}(-1)^{\sigma_j^{t,R,v}}\right|\le J(v,R),$$

where $J(v,R)$ is the number of points q in $X\cap S_R^{n-1}$ such that q is a critical point of $v^*_{|S(q)\cap S_R^{n-1}}$. By Hardt's theorem [Har], this number $J(v,R)$ is uniformly bounded. Since $|I_v^{t,R}|\le 1$, it is easy to get the result. $\square$

For almost all $v\in S^{n-1}$, the function $v^*_{|\{f_R=t\}\cap S_R^{n-1}}$ is a Morse function. Let $\{q_i^{t,R,v},\ i=1,\dots,m_v^{t,R}\}$ be the set of its critical points. With each point $q_i^{t,R,v}$, one can associate two integers : the Morse index of $v^*_{|\{f_R=t\}\cap S_R^{n-1}}$ that we will denote by $\nu_i^{t,R,v}$ and the Morse index of $-v^*_{|\{f_R=t\}\cap S_R^{n-1}}$ that we will denote by $\nu_i^{t,R,-v}$. Observe that the sets $\{p_j^{t,R,v}\}$ and $\{p_j^{t,R,-v}\}$ are subsets of $\{q_i^{t,R,v}\}$, that the set $\{\sigma_j^{t,R,v}\}$ is a subset of $\{\nu_i^{t,R,v}\}$ and that $\{\sigma_j^{t,R,-v}\}$ is a subset of $\{\nu_i^{t,R,-v}\}$. At each point $q_i^{t,R,v}$, we have the following decomposition of v :

$$v=\mu(q_i^{t,R,v})\cdot\nabla f(q_i^{t,R,v})+\lambda(q_i^{t,R,v})\cdot q_j^{t,R,v}.$$

Applying Morse Theory to $\{f_R\le t\}\cap S_R^{n-1}$ and to v^*, we find :

$$\chi(\{f_R\le t\}\cap S_R^{n-1}\cap\{v^*\ge 0\})-\chi(\{f_R\le t\}\cap S_R^{n-1}\cap\{v^*=0\})=$$
$$\sum_{\substack{i|v^*>0\\ \mu>0}}(-1)^{\nu_i^{t,R,v}}+K_v^{t,R},$$

where $K_v^{t,R}$ belongs to $\{-1,0,1\}$ and represents the contribution of the possible critical point of $v^*_{|\{f_R<t\}\cap S_R^{n-1}}$ lying in $\{v^*>0\}$. Similarly, we have :

$$\chi(\{f_R\le t\}\cap S_R^{n-1}\cap\{v^*\le 0\})-\chi(\{f_R\le t\}\cap S_R^{n-1}\cap\{v^*=0\})=$$

$$\sum_{\substack{i|v^*<0\\ \mu<0}} (-1)^{\nu_i^{t,R,-v}} + K_{-v}^{t,R}.$$

Summing these two equalities, we get

$$\chi(\{f_R \leq t\} \cap S_R^{n-1}) - \chi(\{f_R \leq t\} \cap S_R^{n-1} \cap \{v^* = 0\}) = \sum_{\substack{i|v^*>0\\ \mu>0}} (-1)^{\nu_i^{t,R,v}} + \sum_{\substack{i|v^*<0\\ \mu<0}} (-1)^{\nu_i^{t,R,-v}} + K_v^{t,R} + K_{-v}^{t,R}. \quad (1)$$

Furthermore, we have

$$\chi(\{f_R \leq t\} \cap S_R) = \sum_{i|\mu>0} (-1)^{\nu_i^{t,R,v}} + L_v^{t,R}, \quad (2)$$

where $L_v^{t,R}$ belongs to $\{-1, 0, 1\}$ and represents the contribution of the possible critical points of $v^*_{|\{f_R<t\}\cap S_R^{n-1}}$. Similarly, we have :

$$\chi(\{f_R \leq t\} \cap S_R) = \sum_{i|\mu<0} (-1)^{\nu_i^{t,R,-v}} + L_{-v}^{t,R}. \quad (3)$$

Now the combination $-(1) + (2) + (3)$ leads to :

$$\chi(\{f_R \leq t\} \cap S_R) + \chi(\{f_R \leq t\} \cap S_R \cap \{v^* = 0\}) = \sum_{\substack{i|v^*<0\\ \mu>0}} (-1)^{\nu_i^{t,R,v}} + \sum_{\substack{i|v^*>0\\ \mu<0}} (-1)^{\nu_i^{t,R,-v}} + I_v^{t,R} + I_{-v}^{t,R}.$$

By Corollary 3.10, we know that if v does not belong to $\Sigma(X) \cup \Sigma'(X)$, the points $p_j^{t,R,v}$ are exactly the points $q_i^{t,R,v}$ at which $v^* < 0$ and $\mu > 0$, if R is big enough and t is small enough. In the same way, the points $p_j^{t,R,-v}$ are exactly the points $q_i^{t,R,v}$ at which $v^* > 0$ and $\mu < 0$, if R is big enough and t is small enough. Furthermore, if t is sufficiently small $X \cap S_R^{n-1}$ is a strong deformation retract of $\{f_R \leq t\} \cap S_R^{n-1}$ (see [Dur]), hence $\lim_{t\to 0} \chi(\{f_R \leq t\} \cap S_R^{n-1}) = \chi(X \cap S_R^{n-1})$. For $R >$ sufficiently big, $X \cap S_R^{n-1}$ is the link at infinity of X, that we denote by $Lk^\infty(X)$. Finally we get that

$$\lim_{R\to+\infty} \lim_{t\to 0} \chi(\{f_R \leq t\} \cap S_R^{n-1}) = \chi(Lk^\infty(X)).$$

Similarly we have :

$$\lim_{R\to+\infty} \lim_{t\to 0} \chi(\{f_R \leq t\} \cap S_R^{n-1} \cap \{v^* = 0\}) = \chi(Lk^\infty(X \cap \{v^* = 0\})).$$

Collecting all these informations, we find :

$$\lim_{R\to+\infty} \lim_{t\to 0} \left[\sum_{j=1}^{l_v^{t,R}} (-1)^{\sigma_j^{t,R,v}} + I_v^{t,R}\right] + \lim_{R\to+\infty} \lim_{t\to 0} \left[\sum_{j=1}^{l_{-v}^{t,R}} (-1)^{\sigma_j^{t,R,-v}} + I_{-v}^{t,R}\right]$$
$$= \chi(Lk^\infty(X)) + \chi(Lk^\infty(X \cap \{v^* = 0\})).$$

Let us denote $\lim_{R\to+\infty}\lim_{t\to 0}\sum_{j=1}^{l_v^{t,R}}(-1)^{\sigma_j^{t,R,v}}+I_v^{t,R}$ by $\alpha(v)$. We have $\alpha(v)+\alpha(-v)=\chi(Lk^\infty(X))+\chi(Lk^\infty(X\cap\{v^*=0\}))$. But it is easy to see that $\int_{S^{n-1}}\alpha(v)dv=\int_{S^{n-1}}\alpha(-v)dv$, which gives that :

$$\int_{S^{n-1}}\alpha(v)dv=\frac{1}{2}\int_{S^{n-1}}\left[\alpha(v)+\alpha(-v)\right]dv=$$
$$\frac{1}{2}\mathrm{Vol}(S^{n-1})\cdot\chi(Lk^\infty(X))+\frac{1}{2}\int_{S^{n-1}}\chi(Lk^\infty(X\cap\{v^*=0\}))dv.$$

Combining all these results, we can state the Gauss-Bonnet formula for closed semi-algebraic sets.

Theorem 4.5. *Let $X\subset\mathbb{R}^n$ be a closed semi-algebraic set. We have :*

$$\lambda_0(X)=\chi(W)-\frac{1}{2}\chi(Lk^\infty(X))$$
$$-\frac{1}{2\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}}\chi(Lk^\infty(X\cap\{v^*=0\}))dv.$$

□

5. Gauss-Bonnet formula by approximation

In this section, we prove a Gauss-Bonnet theorem for a closed semi-algebraic set by approximation. Our strategy is to approach the semi-algebraic set by a family of closed semi-algebraic sets which are also manifolds with boundary of class C^2 and to integrate the Gauss curvature on the boundaries of these manifolds.

Let us recall the results about semi-algebraic neighborhoods that we obtained in [Dut4]. Let $X\subset\mathbb{R}^n$ be a closed semi-algebraic set and let $\rho:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ be a proper C^2 semi-algebraic function. For every C^2 semi-algebraic function $g:\mathbb{R}^n\to\mathbb{R}$, let $\Gamma_{g,\rho}$ be the semi-algebraic set defined by:

$$\Gamma_{g,\rho}=\{x\in\mathbb{R}^n\mid\nabla g(x)\text{ and }\nabla\rho(x)\text{ are colinear and }g(x)\neq 0\}.$$

Definition 5.1. *Let $g:\mathbb{R}^n\to\mathbb{R}$ be a C^2 semi-algebraic function. We say that g is ρ-quasiregular if there does not exist any sequence $(x_k)_{k\in\mathbb{N}}$ in $\Gamma_{f,\rho}$ such that $\|x_k\|$ tends to infinity and $|g(x_k)|$ tends to 0.*

In [Dut4], we proved the following proposition.

Proposition 5.2. *Let X be closed semi-algebraic set in $\mathbb{R}^n$. There exists a C^2 non-negative semi-algebraic function $f:\mathbb{R}^n\to\mathbb{R}$ such that $X=f^{-1}(0)$ and f is ρ-quasiregular.*

Proof. By [DM] Corollary C.12, it is possible to find a C^2 semi-algebraic non-negative function $g:\mathbb{R}^n\to\mathbb{R}$ such that $X=g^{-1}(0)$. Let r_0 be the greatest critical value of ρ. For $r\geq r_0$, let us denote by Σ_r be the non-empty compact C^2 semi-algebraic hypersurface $\rho^{-1}(r)$. Let $\beta:]r_0,+\infty[\to\mathbb{R}$ be defined by

$$\beta(r)=\inf\{g(x)\mid x\in\Sigma_r\cap\Gamma_{g,\rho}\}.$$

The function β is semi-algebraic. It is positive since for $r > r_0$, the function $g_{|\Sigma_r}$ admits a finite number of critical values. Hence the function $1/\beta$ is semi-algebraic as well. Proposition 2.11 in [Co] (or Proposition 2.6.1 in [BCR]) implies that there exists $r_1 > r_0$ and an integer q_0 such that for all $r \in [r_1, +\infty[$,

$$\frac{1}{\beta(r)} \leq r^{q_0} < (1+r)^{q_0}.$$

Therefore, for all $x \in \Gamma_{g,\rho}$ such that $\rho(x) \geq r_1$ and for all $q \geq q_0$, one has $(1+\rho(x))^q.g(x) > 1$. Since $\Gamma_{g,\rho}$ is equal to $\Gamma_{(1+\rho)^q \cdot g,\rho}$, one can take for f a function $(1+\rho)^q.g$ where q is an integer greater or equal to q_0. $\square$

In order to prove the Gauss-Bonnet formula by approximation, we need an improved version of the previous proposition. Before stating it, let us give a notation. For every $v \in S^{n-1}$ and for every function $f : \mathbb{R}^n \to \mathbb{R}$, f_v will denote the restriction on f to the hyperplane $\{v^* = 0\}$.

Proposition 5.3. *Let X be closed semi-algebraic set in $\mathbb{R}^n$. There exists a C^2 non-negative semi-algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $X = f^{-1}(0)$, f is ρ-quasiregular and for every $v \in S^{n-1}$, f_v is ρ_v-quasiregular.*

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^2 semi-algebraic non-negative function such that $X = g^{-1}(0)$. For every $v \in S^{n-1}$, let r_v be the greatest critical value of ρ_v. Let A be the set defined by :

$$A = \{(v,r) \in S^{n-1} \times \mathbb{R} \mid r > r_v\}.$$

The following set B is semi-algebraic :

$$B = \{(x,v,r) \in \mathbb{R}^n \times S^{n-1} \times \mathbb{R} \mid v^*(x) = 0, \nabla\rho(x) \text{ and } v \text{ are colinear and } r \leq \rho(x)\}.$$

If $\pi : \mathbb{R}^n \times S^{n-1} \times \mathbb{R} \to S^{n-1} \times \mathbb{R}$ is the projection on the last two components, then the set $\pi(B)$ is also semi-algebraic. Since A is the complement of $\pi(B)$, it is semi-algebraic. Let $\gamma : A \to \mathbb{R}$ be defined by :

$$\gamma(v,r) = \inf\{g(x) \mid x \in \Sigma_r \cap \{v^* = 0\} \cap \Gamma_{g_v,\rho_v}\}.$$

A parametric version of Proposition 2.11 in [Co] (see [Mill]) tells us that there exists an integer q_1 such that for every $v \in S^{n-1}$, there exists $r'_v > r_v$ such that for every $r \in [r'_v, +\infty[$:

$$\frac{1}{\gamma(v,r)} \leq r^{q_1} < (1+r)^{q_1}.$$

Then it is easy to conclude the proof of the proposition. $\square$

Using Theorem 3.2 in [Dut4], we obtain the following corollary.

Corollary 5.4. *Let X be closed semi-algebraic set in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 non-negative semi-algebraic function such that $X = f^{-1}(0)$, f is ρ-quasiregular and for every $v \in S^{n-1}$, f_v is ρ_v-quasiregular. Then X is a strong deformation retract of $f^{-1}([0,\delta])$ for $\delta > 0$ sufficiently small and for every $v \in S^{n-1}$, $X \cap \{v^* = 0\}$ is a strong deformation retract of $f^{-1}([0,\delta]) \cap \{v^* = 0\}$ for $\delta > 0$ sufficiently small.*

□

We are in position to state the main result of this section.

Theorem 5.5. *Let X be closed semi-algebraic set in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 non-negative semi-algebraic function such that $X = f^{-1}(0)$, f is ρ-quasiregular and for every $v \in S^{n-1}$, f_v is ρ_v-quasiregular. For every positive regular value δ of f, let k_δ be the curvature of $f^{-1}(\delta)$. We have :*

$$\lim_{\delta\to 0^+}\int_{f^{-1}(\delta)} k_\delta dx = \mathrm{Vol}(S^{n-1})\cdot\chi(X) - \frac{1}{2}\mathrm{Vol}(S^{n-1})\cdot\chi(Lk^\infty(X))$$
$$-\frac{1}{2}\int_{S^{n-1}}\chi(Lk^\infty(X\cap\{v^*=0\}))dv.$$

Proof. Let us study first the case n odd. Theorem 4.5 in [Dut3] says that

$$\int_{f^{-1}(\delta)} k_\delta dx = \frac{1}{2}\mathrm{Vol}(S^{n-1})\cdot\chi(f^{-1}(\delta)) - \frac{1}{2}\int_{S^{n-1}}\chi(f^{-1}(\delta)\cap\{v^*=0\}))dv.$$

One should mention here that in [Dut3], we proved this result only for polynomial functions. To be able to apply it in the semi-algebraic case, we just have to prove that Proposition 3.1 in [Dut3] still holds for a C^1 semi-algebraic function. This is the case because Proposition 3.1 in [Dut3] is a special case of our Lemma 3.5. Since we work in the semi-algebraic setting, Hardt's theorem tells us that $\chi(f^{-1}(\delta)\cap\{v^*=0\})$ is uniformly bounded. Applying Lebesgue's theorem, we get :

$$\lim_{\delta\to 0^+}\int_{f^{-1}(\delta)} k_\delta dx = \mathrm{Vol}(S^{n-1})\cdot\lim_{\delta\to 0^+}\chi(f^{-1}(\delta))$$
$$-\frac{1}{2}\int_{S^{n-1}}\lim_{\delta\to 0^+}\chi(f^{-1}(\delta)\cap\{v^*=0\}))dv.$$

For every $R>0$, we will denote by D_R the set $\{x\in\mathbb{R}^n \mid \rho(x)\le r\}$. For R sufficiently big, it is a C^2 submanifold, with boundary Σ_R, diffeomorphic to a disc. Let us choose $R_0>0$ such that for every $R\ge R_0$, $X\cap D_R^n$ is a deformation retract of X. Since f is ρ-quasiregular, there exits $R_1>0$ such that for every $x\in\Gamma_{f,\rho}$ with $\rho(x)\ge R_1$, one has $f(x)>1$. This implies that for $\delta>0$ sufficiently small and for $R\ge R_1$, $f^{-1}(\delta)\cap D_R$ (resp. $\{f\le\delta\}\cap D_R$) is a deformation retract of $f^{-1}(\delta)$ (resp. $\{f\le\delta\}$). One has

$$\chi(X)=\chi(X\cap D_R)=\lim_{\delta\to 0^+}\chi(\{f\le\delta\}\cap D_R),$$

and

$$\chi(Lk^\infty(X))=\chi(X\cap\Sigma_R)=\lim_{\delta\to 0^+}\chi(\{f\le\delta\}\cap\Sigma_R).$$

For δ small enough, $\{f\le\delta\}\cap D_R$ is a manifold with corners of odd dimension. Therefore, following the method explained in [Dut1], Theorem 5.2, we have :

$$\chi(\{f\le\delta\}\cap D_R)=\frac{1}{2}\chi(\{f=\delta\}\cap D_R)+\frac{1}{2}\chi(\{f\le\delta\}\cap\Sigma_R).$$

Combining all these equalities, we obtain :

$$\frac{1}{2}\lim_{\delta\to 0^+}\chi(f^{-1}(\delta)) = \chi(X) - \frac{1}{2}\chi(Lk^\infty(X)).$$

Now let us fix $v \in S^{n-1}$ and let us set $H = \{x \in \mathbb{R}^n \mid v^*(x) = 0\}$. Since f_v is ρ_v-quasiregular, we can find $R > 0$ sufficiently big such that $X \cap H \cap D_R$ is a deformation retract of $X \cap H$ and such that $f^{-1}(\delta) \cap H \cap D_R$ (resp. $\{f \leq \delta\} \cap H \cap D_R$) is a deformation retract of $f^{-1}(\delta) \cap H$ (resp. $\{f \leq \delta\} \cap H$) for $\delta > 0$ small enough. Since $f^{-1}(\delta) \cap H$ is an odd-dimensional manifold for δ small enough, we can write :

$$\chi(f^{-1}(\delta) \cap H) = \chi(f^{-1}(\delta) \cap H \cap D_R) =$$
$$\frac{1}{2}\chi(f^{-1}(\delta) \cap H \cap \Sigma_R) = \chi(\{f \leq \delta\} \cap H \cap \Sigma_R).$$

Hence, we get

$$\lim_{\delta\to 0^+}\chi(f^{-1}(\delta) \cap H) = \chi(X \cap H \cap \Sigma_R) = \chi(Lk^\infty(X \cap H)).$$

This ends the proof of the case n odd.

If n is even, Theorem 4.5 in [Dut3] states that

$$\int_{f^{-1}(\delta)} k_\delta dx = -\frac{1}{2}\mathrm{Vol}(S^{n-1}) \cdot [\chi(\{f \geq \delta\}) - \chi(\{f \leq \delta\})]$$
$$+\frac{1}{2}\int_{S^{n-1}} [\chi(\{f \geq \delta\} \cap \{v^* = 0\}) - \chi(\{f \leq \delta\} \cap \{v^* = 0\})]\, dv.$$

Applying Lebesgue's theorem, we get :

$$\lim_{\delta\to 0^+}\int_{f^{-1}(\delta)} k_\delta dx = -\frac{1}{2}\mathrm{Vol}(S^{n-1}) \cdot \lim_{\delta\to 0^+}[\chi(\{f \geq \delta\}) - \chi(\{f \leq \delta\})]$$
$$+\frac{1}{2}\int_{S^{n-1}} \lim_{\delta\to 0^+}[\chi(\{f \geq \delta\} \cap \{v^* = 0\}) - \chi(\{f \leq \delta\} \cap \{v^* = 0\})]\, dv.$$

By the Mayer-Vietoris sequence, we have :

$$1 = \chi(\{f \geq \delta\}) + \chi(\{f \leq \delta\}) - \chi(f^{-1}(\delta)),$$

hence,

$$\chi(\{f \geq \delta\}) - \chi(\{f \leq \delta\}) = 1 - 2\chi(\{f \leq \delta\}) + \chi(f^{-1}(\delta)).$$

Choosing $R > 0$ as in the case n odd, we find that

$$\chi(f^{-1}(\delta)) = \chi(f^{-1}(\delta) \cap B_R) = \frac{1}{2}\chi(f^{-1}(\delta) \cap \Sigma_R) = \chi(\{f \leq \delta\} \cap \Sigma_R).$$

Finally, we obtain :

$$\lim_{\delta\to 0^+}[\chi(\{f \geq \delta\}) - \chi(\{f \leq \delta\})] = 1 - 2\chi(X) + \chi(Lk^\infty(X)).$$

Now let us fix $v \in S^{n-1}$ and $H = \{x \in \mathbb{R}^n \mid v^*(x) = 0\}$. We will also use the same R as in the case n odd. As above, we have :

$$\chi(\{f \geq \delta\} \cap H) - \chi(\{f \leq \delta\} \cap H) = 1 - 2\chi(\{f \leq \delta\} \cap H) + \chi(f^{-1}(\delta) \cap H).$$

Moreover, since $\{f \leq \delta\} \cap H$ is odd-dimensional, we can write

$$\chi(\{f \leq \delta\} \cap H \cap D_R) = \frac{1}{2}\chi(f^{-1}(\delta) \cap H \cap D_R) + \frac{1}{2}\chi(\{f \leq \delta\} \cap H \cap \Sigma_R),$$

and therefore,

$$-2\chi(\{f \leq \delta\} \cap H) = -\chi(f^{-1}(\delta) \cap H) - \chi(\{f \leq \delta\} \cap H \cap \Sigma_R).$$

Finally we obtain :

$$\lim_{\delta \to 0^+} [\chi(\{f \geq \delta\} \cap H) - \chi(\{f \leq \delta\} \cap H)] =$$
$$1 - \lim_{\delta \to 0^+} \chi(\{f \leq \delta\} \cap H \cap \Sigma_R) = 1 - \chi(Lk^\infty(X \cap H)).$$

This ends the proof of the case n even. $\square$

Corollary 5.6. *Let X be closed semi-algebraic set in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 non-negative semi-algebraic function such that $X = f^{-1}(0)$, f is ρ-quasiregular and for every $v \in S^{n-1}$, f_v is ρ_v-quasiregular. For every positive regular value δ of f, let k_δ be the curvature of $f^{-1}(\delta)$. We have :*

$$\mathrm{Vol}(S^{n-1}) \cdot \lambda_0(X) = \lim_{\delta \to 0^+} \int_{f^{-1}(\delta)} k_\delta dx.$$

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